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Potential Flow Calculation by the Approximate Factorisation Method

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The approximate factorisation technique for solving the potential flow equation is studied and a condition is derived for determining the form the factorisation should take in a transformed coordinate system. An extension of this scheme to compute transonic flow by solving the nonconservative form of the potential equation is described. Several results are then presented for the calculation of transonic lifting flow over an aerofoil. The convergence of the approximate factorisation algorithm described in this paper is compared with the corresponding behaviour of a successive line overrelaxation method for the identical flow problem. The improvement in convergence rate shown by these numerical comparisons indicates that approximate factorisation can achieve converged results between five and ten times faster than the successive line overrelaxation methods hitherto used to compute potential flow.

INTRODUCTION

Successive line overrelaxation has proved to be a reliable if somewhat slow method for obtaining numerical solutions of the compressible potential equation. The use of mesh refinement does speed convergence but there are still many examples for which relaxation requires a considerable number of iterations. This is indicated by the length of time taken to reach the final converged values of various flow characteristics (e.g., shock position, shock strength and, in the case of a lifting aerofoil, the circulation). The belief that if should be possible to solve such problems more quickly has stimulated the development of several techniques aimed at achieving a significant improvement in the convegence rate. Extrapolation [1], for example, has been successful in accelerating the convergence of relaxation for some flow problems but is apparently limited to those cases where a dominant eigenvalue can be extracted. Methods based on the Poisson solver [2] have been proposed but these do not attain the required improvement in convergence rate for transonic flows with strong shocks. By combining a Poisson solver with a relaxation method, Jameson [3] has overcome this limitation and achieved fast convergence for the lifting aerofoil problem. This particular approach, however, is still limited to a narrow range of problems, typically those two dimensional problems for which the computing mesh can be obtained by a conformal transformation.

Approximate factorisation has now emerged as a particularly promising candidate

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for speeding up flow calculations. The work of Ballhaus *et al.* [4] showed that this technique can achieve substantial improvements in the convergence rates of two dimensional transonic small perturbation (TSP) calculations when compared with the corresponding relaxation solutions. This success encouraged the expectation that a similar improvement could be forthcoming for other flow problems and recent progress by Holst [5, 6] has led to the development of a very successful algorithm for the full potential equation. His method has proved to be extremely effective and rapid convergence has been obtained for a range of difficult flow conditions.

Holst's version of the AF2 factorisation was developed for the conservative form of the full potential equation. In principle, the same factorisation could be used to solve the potential equation in nonconservative form. It seems plausible, though, that an alternative factorisation specifically tailored to the quasilinear equation should be more suitable for this purpose. Work aimed at solving the nonconsevative form of the potential equation by an approximate factorisation technique has led to the development of a factorisation known as AF3 [7]. Although this factorisation was obtained independently of Holst's work, it is essentially the same as his AF2 scheme when applied to incompressible flow. For compressible flow and particularly in their treatment of supersonic regions, these two schemes differ.

In this paper we first develop some general results relating to schemes of the AF2 or AF3 type. This leads to a discussion of the form the factorisation should take under different coordinate transformations. Finally, the extension of the AF3 algorithm for computing transonic flow is described and several results are presented for the calculation of transonic lifting flow over an aerofoil.

Approximate Factorisation Schemes

We start by considering the general iterative scheme

$$N\Delta^n_{i,j} = L\phi^n_{i,j}, \tag{1}$$

where L is a finite difference operator such that $L\phi^n_{i,j}$ represents a finite difference approximation to the partial differential equation we wish to solve. We call $L\phi^n_{i,j}$ the residual after cycle n. As an example we take Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0$$

and form the usual second order accurate central difference approximation

$$L\phi^n_{i,j} = \left(\frac{\vec{\delta}_x \overleftarrow{\delta}_x}{\Delta x^2} + \frac{\vec{\delta}_y \overleftarrow{\delta}_y}{\Delta y^2}\right)\phi^n_{i,j}. \tag{2}$$

The second order central difference operator $\vec{\delta}_x \overleftarrow{\delta}_x$ has been formed by a combination of the first order forward and backward difference operators in the x direction. These are defined by

$$\vec{\delta}_x \phi_{ij} = \phi_{i+1,j} - \phi_{ij},$$
$$\overleftarrow{\delta}_x \phi_{i,j} = \phi_{ij} - \phi_{i-1,j}.$$

We define the correction vector for cycle $n+1$ as

$$\Delta_{ij}^n = \phi_{ij}^{n+1} - \phi_{ij}^n$$

which can be interpreted as a forward difference operator if one considers the iteration cycles as steps in artificial time. If we make this identification then Eq. (1) can be regarded as a finite difference approximation to a time dependent equation. The operator N has to be selected so that the iterative scheme is stable and the residual on the right hand side of Eq. (1) is driven towards zero as quickly as possible. In terms of the time dependent analogy this means that we wish to approach the steady state solution as quickly as possible. It has been shown [4] that N should closely resemble the operator L in order to achieve convergence is as few iterations as possible. On the other hand, N should have a fairly simple form and be easy to invert in order to keep the computation time for one iteration cycle reasonably low. Approximate factorisation schemes seek a compromise between these two conflicting requirements by splitting N into a number of simple factors

$$N = N_1 N_2 \cdots N_k.$$

As an example of this idea we consider the following ADI or AF1 scheme

$$\left(\alpha - \frac{\vec{\delta}_x \overleftarrow{\delta}_x}{\Delta x^2}\right)\left(\alpha - \frac{\vec{\delta}_y \overleftarrow{\delta}_y}{\Delta y^2}\right) \Delta_{ij}^n = 2\alpha L\phi_{ij}^n, \tag{3}$$

where the residual on the right hand side is given by Eq. (2). If we write $\alpha = 2/\Delta t$, the difference scheme (3) can be regarded as a second order accurate finite difference approximation to the parabolic equation

$$\phi_t = \phi_{xx} + \phi_{yy}.$$

The presence of the ϕ_t term has the effect of damping any transient solutions and thus helps to ensure that the iterative scheme (3) is convergent. If, however, we consider the solution of an equation which is hyperbolic in character then it can be shown [8] that the ϕ_t term has a destabilising effect. Its presence allows solutions that increase exponentially with time and these would quickly swamp the desired steady state solution. It follows that the ADI or AF1 schemes, although very efficient for solving elliptic and parabolic equations, are not suitable for the solution of an equation of mixed type. Since we are interested in solving the potential equation which is of mixed type in a transonic flow condition, we need an alternative factorisation that does not give rise to a ϕ_t term in the associated time dependent equation.

If instead of the AF1 scheme (3) we split one of the second order central difference operators between the two factors, we obtain a factorisation of the AF2 or AF3 type [4–7]. For example, splitting in the y coordinate direction leads to the scheme

$$\left(-\alpha\frac{\vec{\delta}_y}{\Delta y}-\frac{\vec{\delta}_x\overleftarrow{\delta}_x}{\Delta x^2}\right)\left(\alpha+\frac{\overleftarrow{\delta}_y}{\Delta y}\right)\Delta_{ij}^n=\alpha L\phi_{ij}^n, \tag{4}$$

where we again assume that the residual is given by Eq. (2). The time dependent equation associated with this iterative scheme now has the desired property of containing no ϕ_t term.

Without loss of generality we assume that

$$\Delta x=\Delta y=h.$$

A von Neumann analysis shows that the iterative scheme (4) is stable and that the sequence of acceleration parameters,

$$\alpha=\frac{2\sin ph/2}{h},\qquad p=1\cdots\frac{\pi}{h}, \tag{5}$$

approximately minimises the amplification factor. This analysis holds for a uniform Cartesian mesh but under a transformation to a stretched coordinate system the conclusion is no longer valid. It is then necessary to modify either the parameter sequence (5) or preferably the form of the factorisation (4) in order to attain fast convergence on a stretched mesh. This observation was brought to the author's attention by Dr. Catherall who has successfully applied approximate factorisation to calculate transonic flow over an aerofoil on a non-aligned mesh [9], (i.e., a mesh in which the body surface is not aligned with any particular coordinate line).

Consider the effect of introducing a coordinate transformation to a new set of orthogonal curvilinear coordinates $X(x, y)$ and $Y(x, y)$ and write

$$C_1\phi_{XX}+C_2\phi_{YY} \tag{6}$$

for the second order derivative terms that then appear in Laplace's equation. In place of the factorisation (4) we consider the general form

$$\left(-\alpha A_2\frac{\vec{\delta}_Y}{\Delta Y}-A_1\frac{\vec{\delta}_X\overleftarrow{\delta}_X}{\Delta X^2}\right)\left(B_1\alpha+B_2\frac{\overleftarrow{\delta}_Y}{\Delta Y}\right)\Delta_{ij}^n=\alpha L\phi_{ij}^n, \tag{7}$$

where A_1, A_2, B_1 and B_2 are functions of X and Y which satisfy the conditions

$$A_1B_1=C_1\qquad\text{and}\qquad A_2B_2=C_2 \tag{8}$$

but are otherwise arbitrary.

We now seek a relationship between these four coefficients that will achieve the most rapid convergence. The von Neumann analysis that we use is only strictly correct when the coefficients are constant. However, we assume that the variation is small over distances of the order of a mesh width and hence that the analysis is approximately valid locally.

Let the error after the nth iterative cycle be expressed as

$$e^n = \sum_{p,q} \rho^n(p,q)\, e^{ipX} e^{iqY}$$

and consider the effect of scheme (7) on the particular error component corresponding to the frequency pair (p, q). Again we assume that $\Delta X = \Delta Y = h$ and write

$$S_p = \frac{\sin ph/2}{h/2} \qquad \text{and} \qquad e_p = e^{iph/2}.$$

We find that the amplification factor is

$$G(p,q) \equiv \frac{\rho^{n+1}}{\rho^n} = \frac{-i\alpha^2 A_2 B_1 S_q e_q + iA_1 B_2 S_q S_p^2 e_q^*}{(A_1 S_p^2 - iA_2 \alpha S_q e_q)(B_1 \alpha + iB_2 S_q e_q^*)}.$$

After dividing through by $A_1 B_1$ and writing $A = A_2/A_1$, $B = B_2/B_1$, the following expression is obtained:

$$|G|^2 = \frac{S_q^2[\alpha^4 A^2 - 2\alpha^2 AB S_p^2 \cos qh + B^2 S_p^4]}{[S_p^4 + h\alpha A S_q^2 S_p^2 + \alpha^2 A^2 S_q^2][\alpha^2 + h\alpha B S_q^2 + B^2 S_q^2]}. \tag{9}$$

The combination

$$AB = \frac{A_2 B_2}{A_1 B_1} = \frac{C_2}{C_1}$$

is a constant for the purposes of our stability analysis. For the case

$$A_1 = B_1 = A_2 = B_2 = 1$$

we know from (5) that the choice $\alpha = |S_p|$ approximately minimises $|G(p,q)|^2$. If we substitute this value of α into (9) we then obtain

$$|G|^2 = \frac{S_q^2 S_p^2[A^2 - 2AB\cos qh + B^2]}{[S_p^2 + Ah|S_p|S_q^2 + A^2 S_q^2][S_p^2 + Bh|S_p|S_q^2 + B^2 S_q^2]} \tag{10}$$

for the square modulus of the amplification factor corresponding to the general scheme (7).

It can readily be seen that an inappropriate choice of the coefficients A_1, A_2, B_1 and B_2 can cause slow convergence. For example, if we take the extreme case

$$\frac{A_2}{A_1} \gg 1 \gg \frac{B_2}{B_1}$$

in expression (10) we find that

$$|G| \sim 1.$$

It follows that a transformation from Cartesian coordinates to a stretched coordinate system can have an adverse effect on the convergence rate of an approximate factorisation scheme.

We now determine the relationship between A and B which will minimise $|G|^2$ subject to the constraint that AB is constant. Expression (10) is symmetric in A and B and our problem can be restated as finding the minimum of the function

$$\Gamma(x) = \frac{g(x)}{r(x)\, r(k/x)},$$

where

$$r(x) = ax^2 + bx + c$$

and

$$g(x) = x^2 - 2k\mu + k^2/x^2, \qquad \mu \leqslant 1.$$

To obtain the above expression the substitutions

$$k = AB, \qquad x = A,$$
$$a = S_q^2, \qquad b = h\,|S_p|\, S_q^2, \qquad c = S_p^2$$

have been made.

Now

$$\Gamma'(x) = \frac{g'(x)\, r(x)\, r(k/x) - g(x)\{r'(x)\, r(k/x) - k/x^2\, r'(k/x)\, r(x)\}}{r(x)^2\, r(k/x)^2}$$

with $g'(x) = 2x(1 - k^2/x^4)$ and $r'(x) = 2ax + b$. When $x = \sqrt{k}$ we obtain $g'(\sqrt{k}) = 0$ and hence

$$\Gamma'(\sqrt{k}) = 0.$$

Thus $\Gamma(x)$ has a stationary value at the point $x = \sqrt{k}$. To determine whether this is a minimum we evaluate the second derivative $\Gamma''(x)$ at $x=\sqrt{k}$ and obtain the following expression:

$$\Gamma''(\sqrt{k})=\frac{g''(\sqrt{k})}{r(\sqrt{k})^2}-\frac{2g(\sqrt{k})[r(\sqrt{k})\,r''(\sqrt{k})-r'(\sqrt{k})^2+r(\sqrt{k})\,r'(\sqrt{k})/\sqrt{k}]}{r(\sqrt{k})^4}.$$

The point $x=\sqrt{k}$ will be a minimum if this expression is positive. That is, if

$$g''(\sqrt{k})\,r(\sqrt{k})^2>2g(\sqrt{k})[r(\sqrt{k})\,r''(\sqrt{k})-r'(\sqrt{k})^2+r(\sqrt{k})\,r'(\sqrt{k})/\sqrt{k}].$$

Now $g''(\sqrt{k})=8$ and $g(\sqrt{k})=2k(1-\mu)\leqslant 4k$. It follows that the above inequality holds if

$$r(\sqrt{k})^2>k[r(\sqrt{k})\,r''(\sqrt{k})-r'(\sqrt{k})^2+r(\sqrt{k})\,r'(\sqrt{k})/\sqrt{k}]. \tag{11}$$

On expanding out the terms on the right hand side we find that

$$\begin{aligned}&k[r(\sqrt{k})\,r''(\sqrt{k})-r'(\sqrt{k})^2+r(\sqrt{k})\,r'(\sqrt{k})/\sqrt{k}]\\&\quad=abk\sqrt{k}+4ack+bc\sqrt{k}.\end{aligned}$$

But

$$\begin{aligned}r(\sqrt{k})^2&=(ak+b\sqrt{k}+c)^2\\&=a^2k^2+2abk\sqrt{k}+b^2k+2ack+2bc\sqrt{k}+c^2\\&=(ak-c)^2+2abk\sqrt{k}+b^2k+4ack+2bc\sqrt{k}\end{aligned}$$

which is certainly greater than the right hand side of (11). It follows that inequality (11) is satisfied and hence $\Gamma''(\sqrt{k})>0$. Thus, the value $x=\sqrt{k}$ minimises the function $\Gamma(x)$.

In other words the amplification factor is minimised by the choice

$$\frac{A_2}{A_1}=\frac{B_2}{B_1}=\sqrt{\frac{C_2}{C_1}}. \tag{12}$$

On making the substitution $A=B$ in expression (10) corresponding to an optimum coefficient choice (12), we find that

$$|G|=\frac{h\,|S_p|\,S_q^2A}{S_p^2+h\,|S_p|\,AS_q^2+A^2S_q^2}.$$

On using the inequality

$$S_p^2+A^2S_q^2\geqslant 2\,|S_p|\,|S_q|\,A$$

we obtain

$$|G| \leqslant \frac{h\,|S_q|}{2 + h\,|S_q|}.$$

But $h\,|S_q| = 2\,|\sin(qh/2)| \leqslant 2$ whence we get the bound

$$|G| \leqslant \tfrac{1}{2}.$$

Thus, when the coefficients are chosen according to (12) and $\alpha = |S_p|$, the pth wave component is reduced in magnitude by at least one-half. In order to reduce all wave components we use the sequence of acceleration parameters given by (5). In practice, a small number of parameters will usually suffice provided they cover the range between the maximum and minimum (i.e., between $2/h$ and 1) of the sequence (5). If we therefore take a sequence of say, six acceleration parameters, then all wave components and hence the complete error vector e^n should be reduced by at least one-half every six iterations. Thus if we define the average convergence rate as

$$\lim_{n\to\infty} \left(\frac{e^n}{e^0}\right)^{1/n}$$

then this is bounded from above by

$$(\tfrac{1}{2})^{1/6} \simeq 0.89.$$

This ties in well with the observed residual reduction rate of about 0.87 which corresponds to a reduction of the residual by three orders of magnitude every 50 iterations.

The AF3 Scheme for Mixed Flow

Condition (12) is a useful guide to the construction of a successful approximate factorisation method. In this section we extend the factorisation to treat mixed flow conditions by introducing upwind differencing when the flow is locally supersonic. In order to simplify the presentation we first assume that the computational space has been obtained by a conformal mapping. The extension to a more general coordinate system then follows in a straightforward manner.

Consider the conformal transformation

$$Z = f(z),$$

where $Z = X + iY$ is a point in the computational space and $z = x + iy$ represents the corresponding point in physical space. If the mapping modulus is

$$H = \left|\frac{dZ}{dz}\right|$$

then the coefficients that appear in (6) are

$$C_1 = C_2 = H^2.$$

Condition (12) therefore reduces to

$$\frac{A_2}{A_1} = \frac{B_2}{B_1} = 1$$

which implies that $A_1 = A_2$ and $B_1 = B_2$. It follows that the factorisation should take the form

$$\left(-\alpha A \frac{\vec{\delta}_Y}{\Delta Y} - A \frac{\vec{\delta}_X \overleftarrow{\delta}_X}{\Delta X^2}\right)\left(B\alpha + B \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right) \Delta^n_{ij} = \alpha L^n_{ij}$$

with $AB = H^2$. If we now expand out this factorisation, the operators appearing in the first factor will act on B to give terms like

$$-\alpha^2 A \frac{\partial B}{\partial Y} - \alpha A \frac{\partial^2 B}{\partial X^2}.$$

When B is a function of X and Y, these extra terms will be non-zero and may well have an adverse effect on the convergence rate of the iterative scheme. In any case they introduce a ϕ_t term in the associated time dependent equation and can therefore be expcted to cause instability in a region of supersonic flow. We therefore make the restriction that the coefficient of α in the second factor is a constant which, without loss of generality, we may take equal to unity. It follows that $B = 1$ and hence $A = H^2$ so that the best approximate factorisation scheme for use on a mesh produced by a conformal mapping is

$$\left(-\alpha H^2 \frac{\vec{\delta}_Y}{\Delta Y} - H^2 \frac{\vec{\delta}_X \overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right) \Delta^n_{ij} = \sigma\alpha L\phi^n_{ij}. \tag{14}$$

In Eq. (14) we have introduced a relaxation factor σ. It is found that a value for σ slightly greater than one improves the convergence speed of this approximate factorisation scheme. A value of about 1.3 is generally used. The acceleration parameters can be chosen according to (5) but in practice the following parameter sequence [4] has proved to be very effective:

$$\alpha = \alpha_h \left(\frac{\alpha_l}{\alpha_h}\right)^{(k-1)/(N-1)}, \qquad k = 1 \cdots N,$$

where $\alpha_l = 1$ and $\alpha_h = 2/h$. A value for N of either 6 or 8 is usually taken for the number of parameters in the sequence which is repeated in a cyclic fashion.

We now apply these ideas to the potential equation in its quasilinear form. In a transformed coordinate system the equation can be written as

$$A\phi_{XX} + B\phi_{XY} + C\phi_{YY} + D = 0. \tag{15}$$

Under the conformal mapping considered above the coefficients are as follows:

$$A = H^2\left(1 - \frac{u^2}{a^2}\right), \qquad B = -2H^2\frac{uv}{a^2}, \qquad C = H^2\left(1 - \frac{v^2}{a^2}\right),$$

where u and v are the velocity components in the X and Y directions, respectively. The sound speed is determined from Bernoulli's equation

$$a^2 = \frac{1}{M_\infty^2} + \frac{(\gamma - 1)}{2}(1 - u^2 - v^2).$$

If we use central difference approximations to represent the various derivatives of ϕ and thus construct a residual $L\phi_{ij}^n$ corresponding to Eq. (15) the factorisation

$$\left(-\alpha C\frac{\overleftarrow{\delta}_Y}{\Delta Y} - A\frac{\overrightarrow{\delta}_X\overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right)\Delta_{ij}^n = \sigma\alpha L\phi_{ij}^n \tag{16}$$

can be used to solve the equation provided the flow is subcritical.

When regions of supersonic flow are present, computational stability requires the introduction of a rotated difference scheme [8]. This involves a combination of centrally differenced and upwind differenced second derivative terms chosen so that the numerical domain of dependence contains that of the differential equation. We therefore write

$$A = A_u + A_c, \qquad B = B_u + B_c, \qquad C = C_u + C_c,$$

where the subscript u refers to the contribution of a coefficient to the upwind differenced term and the subscript c denotes the centrally differenced contribution. The residual now contains both central and upwind differenced terms and the factorisation (16) can be modified as follows:

$$\left(-\alpha C_c\frac{\overleftarrow{\delta}_Y}{\Delta Y} - A_c\frac{\overrightarrow{\delta}_X\overleftarrow{\delta}_X}{\Delta X^2} - A_u\frac{\overleftarrow{\delta}_X\overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right)\Delta_{ij}^n = \sigma\alpha L\phi_{ij}^n. \tag{17}$$

This form applies when u, the velocity component in the X direction, is positive; forward differencing is used when u is negative. The upwind differencing in the above factorisation complicates the algorithm and, allowing for the possibility of both backward and forward differencing in the X direction, this requires the inversion of a pentadiagonal matrix. If we drop one of the $\overleftarrow{\delta}_X$ operators we obtain the alternative factorisation

$$\left(-\alpha C_c \frac{\overrightarrow{\delta}_Y}{\Delta Y} - A_c \frac{\overleftarrow{\delta}_X \overleftarrow{\delta}_X}{\Delta X^2} - A_u \frac{\overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right) \Delta^n_{ij} = \sigma \alpha L \phi^n_{ij} \tag{18}$$

which only requires a tridiagonal matrix inversion. Removing one of the $\overleftarrow{\delta}_X$ operators has the effect of replacing the upwind formula

$$(\Delta X)^2 \phi_{XX} = \phi^{n+1}_{ij} - 2\phi^{n+1}_{i-1,j} + \phi^{n+1}_{i-2,j}$$

by

$$(\Delta X)^2 \phi_{XX} = \phi^{n+1}_{ij} - \phi^{n+1}_{i-1,j} - \phi^n_{i-1,j} + \phi^n_{i-2,j}.$$

A stability analysis indicates that this alternative form is stable and a comparison of the two factorisations for a number of numerical examples confirms that the convergence rates are similar. One also finds that the upwind differenced term in (18) introduces a ϕ_{Xt} term in the associated time dependent equation. This is known to have a favourable influence on the stability of the iterative scheme when regions of supersonic flow are present [8].

It should be noticed that the factorisation contains no upwind differenced term in the Y direction. Thus the upwind differencing for ϕ_{YY} is evaluated at the previous level n, viz.,

$$(\Delta Y)^2 \phi_{YY} = \phi^n_{ij} - 2\phi^n_{i,j-1} + \phi^n_{i,j-2}.$$

This does not appear to slow the convergence of the approximate factorisation scheme, presumably because the ϕ_{Xt} term is sufficient to maintain stability. If desired, however, the factorisation can be modified to accommodate a forward difference approximation for ϕ_{YY} when v, the velocity component in the Y direction, is negative, viz.,

$$\left(-\alpha C_c \frac{\overrightarrow{\delta}_Y}{\Delta Y} - \alpha C_u \frac{E_Y \overrightarrow{\delta}_Y}{\Delta Y} - A_c \frac{\overleftarrow{\delta}_X \overleftarrow{\delta}_X}{\Delta X^2} - A_u \frac{\overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \frac{\overleftarrow{\delta}_Y}{\Delta Y}\right) \Delta^n_{ij} = \sigma \alpha L \phi^n_{ij}, \tag{19}$$

where E_Y is the shift operator defined by

$$E_Y \phi_{ij} = \phi_{ij+1}.$$

The AF3 scheme described above has been used to solve the potential equation over a wide range of flow conditions and for various types of coordinate mesh. These include lifting aerofoil calculations where the mesh is generated by a conformal mapping to the inside of a circle [10] and channel flow calculations using a sheared and hence non-orthogonal coordinate system..

Before showing some numerical results we complete our discussion of the AF3 scheme by examining the general case. Although our analysis was restricted to orthogonal transformations we assume that condition (12) still applies when the mesh is mildly non-orthogonal. As before it is necessary for the coefficient of α in the second factor of (7) to be a constant. Thus we take $B_1 = 1$ and since

$$A_1 B_1 = C_1$$

we require $A_1 = C_1$. Condition (12) for optimum convergence then implies that

$$\frac{A_2}{A_1} = \frac{B_2}{B_1} = \sqrt{\frac{C_2}{C_1}}.$$

Hence, $A_2 = \sqrt{C_1 C_2}$ and $B_2 = \sqrt{C_2/C_1}$. Under a general coordinate transformation we therefore require the following factorisation:

$$\left(-\alpha\sqrt{C_1 C_2}\,\frac{\vec{\delta}_Y}{\Delta Y} - C_1\,\frac{\vec{\delta}_X \overleftarrow{\delta}_X}{\Delta X^2}\right)\left(\alpha + \sqrt{\frac{C_2}{C_1}}\,\frac{\overleftarrow{\delta}_Y}{\Delta Y}\right)\Delta^n_{ij} = \sigma\alpha L\phi^n_{ij}.$$

We note that when the transformation is conformal we have

$$C_1 = C_2 = H_2$$

and the factorisation then reduces to the previous case (14). The modification to treat transonic potential flow follows in an essentially identical manner to that described above for the conformal transformation.

Results

We now present some comparisons between a successive line overrelaxation method and the AF3 scheme described above. The calculations are for lifting transonic flow over an aerofoil using the nonconservative form of the potential equation. The coordinate mesh has been obtained by a conformal mapping of the region exterior to the aerofoil onto the inside of a circle [10]. By taking equal increments Δr in the radial direction and $\Delta\theta$ in the circumferential direction we obtain a mesh that is nonuniform in physical space. The relaxation code mentioned here is very similar to the well known Garabedian and Korn method [11] which also uses this coordinate mesh. It follows that the convergence information presented for the relaxation method can be regarded as typical of the behaviour that would be shown by the Garabedian and Korn aerofoil analysis code.

The relaxation solutions were carried out in an optimum manner, first on a coarse mesh and then, using this result as the starting point for the final run, on a fine mesh of 161×32 points. In contrast, all AF calculations were computed on the fine mesh only. In fact, no significant advantage seems to be gained by using mesh refinement with AF. Some numerical experiments were initially carried out to determine the

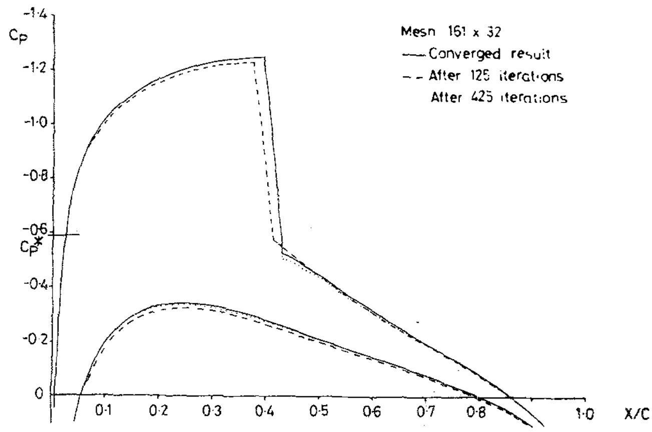

FIG. 1. Relaxation computation of pressure distribution for NACA 0012, $M_\infty = 0.75$, $\alpha = 2°$.

endpoints α_1 and α_h to be used with the sequence of acceleration parameters. In line with the modal analysis a value $\alpha_1 = 1$ proved best. The precise value of α_h does not appear to be critical provided it is of the same order as that indicated by the modal analysis. In these calculations a value for α_h greater than about 10 is appropriate. Increasing α_h causes a slight deterioration in the convergence rate but enhances stability for flows containing strong shocks. A value $\alpha_h = 20$ appears to give both fast and stable convergence over the range of flow conditions normally encountered. All

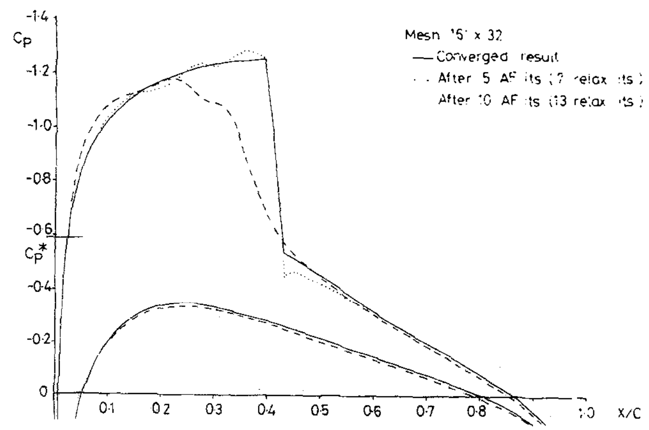

FIG. 2. AF3 computation of pressure distribution for NACA 0012, $M_\infty = 0.75$, $\alpha = 2°$.

AF computations presented here have therefore been computed using the following set of parameters,

$$\alpha = \alpha_h \left(\frac{\alpha_1}{\alpha_h}\right)^{(k-1)/(N-1)}, \qquad k = 1 \cdots N,$$

with $\alpha_1 = 1$, $\alpha_h = 20$, $N = 8$ and with a relaxation factor $\sigma = 1.3$.

The first case we consider is the NACA 0012 aerofoil at a Mach number of 0.75 and an incidence angle of 2°. Figure 1 compares the converged pressure distribution with the computed result after 125 and 425 iterations of the relaxation method. It is evident that the result after 125 iterations is some way off convergence and even after 425 iterations the shock strength has still not reached its final converged value. It is interesting to contrast these results with those shown in Fig. 2 for AF3 after 5 and 10 AF iterations (equivalent in computing time to 7 and 13 relaxation iterations,

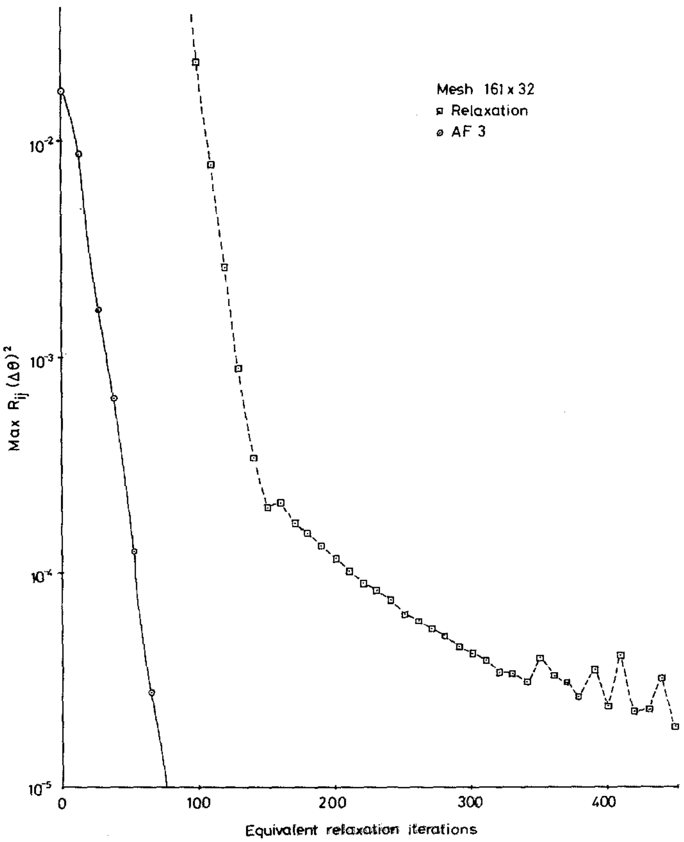

FIG. 3. Convergence histories for NACA 0012, $M_\infty = 0.75$, $\alpha = 2^0$.

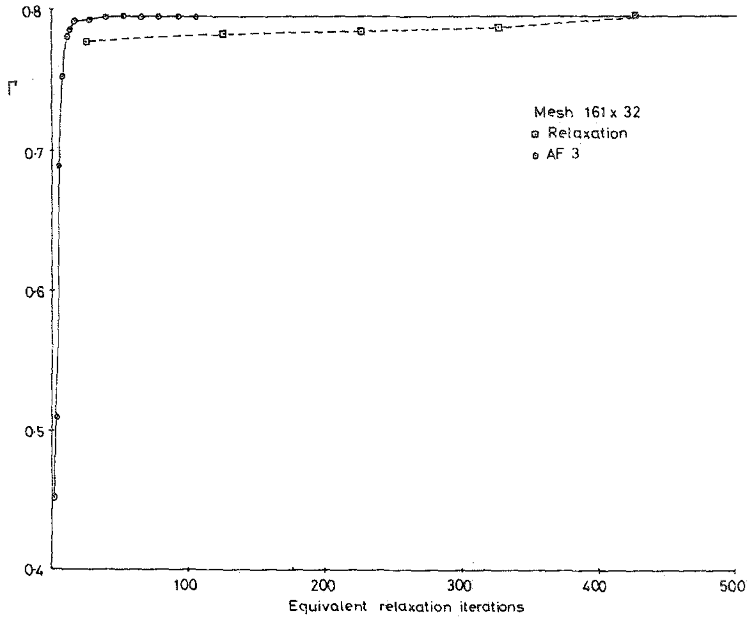

FIG. 4. Growth of circulation Γ for NACA 0012, $M_\infty = 0.75$, $\alpha = 2°$.

respectively). After 10 AF iterations the lower surface distribution is correct and the final shock position has been reached. After a further 10 AF iterations there is no plottable difference between the computed pressure distribution and the converged result.

One measure of the degree of convergence of an iterative scheme is given by the

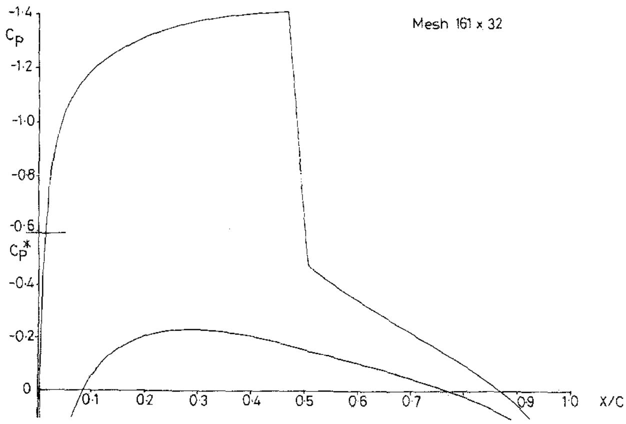

FIG. 5. Pressure distribution for NACA 0012, $M_\infty = 0.75$, $\alpha = 3°$.

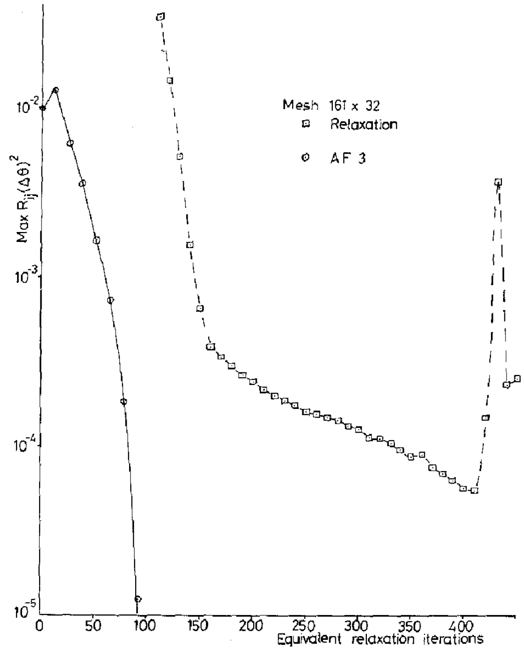

FIG. 6. Convergence histories for NACA 0012, $M_\infty = 0.75$, $\alpha = 3°$.

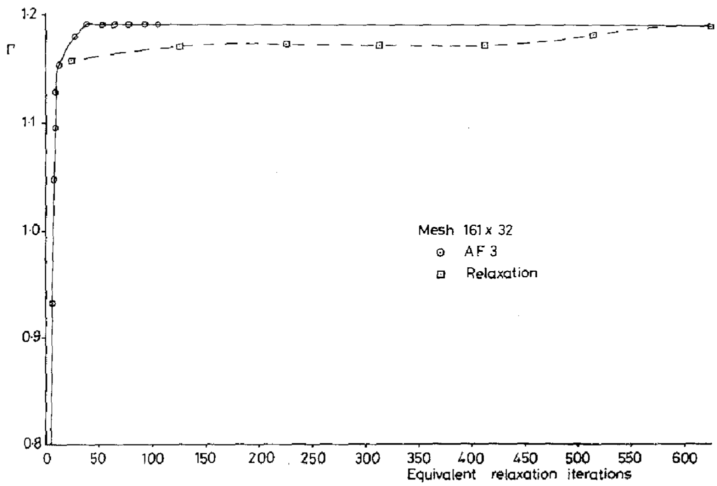

FIG. 7. Growth of circulation Γ for NACA 0012, $M_\infty = 0.75$, $\alpha = 3°$.

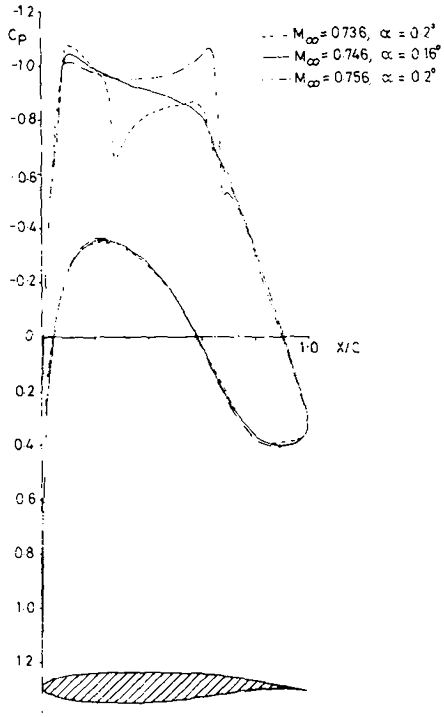

FIG. 8. Pressure distributions for Airfoil 75-06-12 (KORN1).

maximum value of the residual. Complete convergence corresponds to a maximum residual of zero although in practice round-off error sets a lower limit which will depend on the word length of the computer used. We have multiplied the maximum residual by $(\Delta\theta)^2$, where $\Delta\theta = 2\pi/160$ is the increment in the computational variable θ. Values of this convergence parameter are plotted against equivalent relaxation iterations in order to get a sensible comparison between the different methods. Thus, for example, Fig. 3 compares the convergence histories of relaxation and AF3. A value of 10^{-5} for the convergence parameter is usually necessary to guarantee an adequate level of convergence. According to this criterion, relaxation requires about 700 iterations, while AF3 requires the equivalent of only 75 relaxation iterations.

Another indicator of the convergence rate is the growth of circulation towards the final value. Figure 4 presents a comparison of the circulation growth for relaxation and AF3 again plotted in terms of equivalent relaxation iterations. AF3 quickly reaches the converged value in sharp contrast to the relaxation calculation.

Figure 5 shows the pressure distribution for NACA 0012 at a Mach number of 0.75 and an incidence angle of 3°. The presence of a strong shock on the upper surface for this case provides a severe test for any numerical method, but as shown in Fig. 6, AF3 rapidly achieves an adequate level of convergence. Figure 7 shows the

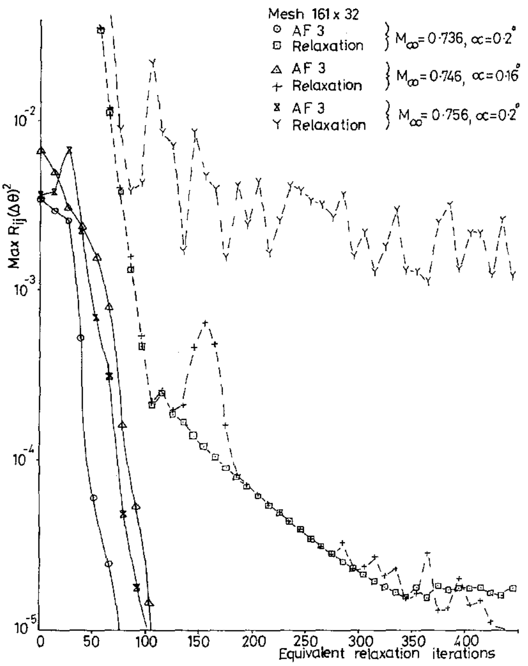

FIG. 9. Convergence histories for Airfoil 75-06-12 (KORN1).

circulation growth for this case and a particularly interesting feature is the behaviour of the relaxation method. Between about 200 and 400 iterations the circulation computed by the relaxation method remains remarkably constant but only reaches the final converged level much later.

Finally, we present some comparisons for Airfoil 75-06-12, otherwise known as KORN1. Figure 8 shows the pressure distribution at an essentially shock free condition and two off design conditions. The convergence histories for AF3 and relaxation are shown in Fig. 9 and the superior performance of the AF scheme is again apparent. The convergence history for the 0.756 Mach number case indicates that convergence of the relaxation method is particularly lengthy and difficult. It is possible that a more judicious choice of the initial potential distribution might lead to better convergence. This dependence of relaxation convergence on a good choice of starting condition is, however, in sharp contrast to the AF scheme which converges equally well from most initial field distributions.

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